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Kadison's Schwarz and Kantorovich inequalities on correlation operators

Saichi Izumino and Masahiro Nakamura

Abstract. Applying Kadison's Schwarz inequality and the Kantorovich inequality to Hadamard products of operators, we show some facts on correlation operators which are defined in virture of the Hadamard product.

1. As observed in the recent standard reference book [5] on the theory of operator inequalities, Kadison's Schwarz inequality and the celebrated Kantorovich inequality play a central role in that theory, whereas they are not refered in the theory of Hadamard products of operators. In the present note we shall commit some facts on the applications of the two important inequalities.

Throughout this note, an operator means a bounded linear operator acting on a Hilbert space H with a complete orthonormal system $\{e_n\}$, and $A * B$ represents the Hadamard product of operators A and B according to the definition of Fujii [3], cf. [6] with respect to the basis $\{e_n\}$.

2. H. Umegaki introduced an expectation $\mathrm{E}[A]$, called the diagonalization of an operator A defined by

$$\mathrm{E}[A] = A * I \quad (I \text{ is the identity operator on } H),$$

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which has been already substancially discussed by Ando [1]. A basic property of the map E is

$$\mathrm{E}[A * B] = \mathrm{E}[\mathrm{E}[A] * B] = \mathrm{E}[A * \mathrm{E}[B]]. \tag{1}$$

Following Styan [7], a correlation operator R is defined as a positive operator such that $R * I = I$ or $\mathrm{E}[R] = I$. Since E is a unital positive linear map on the algebra $B(H)$ of all operators, we have the following theorem from Kadison's Schwarz inequality (cf. [5, Theorem 1.17]):

Theorem 1. *Let R be a correlation operator. Then*

$$R^{-1} * I \geq (R * I)^{-1} = I \quad \textit{if } R \textit{ is invertible}, \tag{2}$$

and

$$R^2 * I \geq (R * I)^2 = I. \tag{3}$$

From (2) we have the following

Corollary 2. *If $\tilde{r}_{i,i}$ are diagonal entries of R^{-1} (with respect to the system $\{e_n\}$), then*

$$\tilde{r}_{i,i} \geq 1. \tag{4}$$

Diagonal entries of R^2 also satisfy similar inequalities.

An application of the Kantorovich inequality (cf. [5, Theorem 1.32 (iv)]) to the diagonalization of operators yields the following fact due to Kitamura and Seo.

Theorem 3 ([6, Corollary 2]). *If R is a correlation operator satisfying $mI \leq R \leq MI$ for some positive scalars $m < M$, then*

$$I \leq R^{-1} * R * I \leq \frac{(M+m)^2}{4Mm} I. \tag{5}$$

From Fiedler's inequality (cf. [5, Theorem 6.8]) we have $R^{-1} * R \geq I$, so that the first inequality is immediate. The second inequality follows from the Kantorovich inequality since $R^{-1} * R * I = R^{-1} * I$.

Similarly as above, from the Kantorovich inequality (cf. [5, Theorem 1. 32 (iii)]) we have

Theorem 4. *If R is a correlation operator satisfying $mI \leq R \leq MI$ for some positive scalars $m < M$, then*

$$R^2 * I \leq \frac{(M+m)^2}{4Mm} I. \tag{6}$$

Proof. By the Kantorovich inequality we have

$$R^2 * I \leq \frac{(M+m)^2}{4Mm}(R * I)^2 = \frac{(M+m)^2}{4Mm} I.$$

□

As an extension of the above inequality (6) the following fact is known (cf. [5, Theorem 6. 16 (iii)]): For positive operators A and B satisfying $0 < mI \leq A,\ B \leq MI$

$$(A^2 * B^2)^{1/2} \leq \frac{M+m}{2\sqrt{Mm}} A * B,$$

which is also a consequence of the Kantorovich inequality.

3. Styan's main result in [7] is the following:

$$(R * R^{-1})\nabla I \geq (R * R)^{-1} \quad \text{for all invertible correlation operators } R, \tag{7}$$

where ∇ means the arithmetic mean.

Related to the above inequality we have

Theorem 5. *If R is an invertible correlation operator, then*

$$R * R^{-1} \geq (R * R)^{-1}. \tag{8}$$

Proof. One of the proof is due to R. Nakamoto: By Fiedler's inequality and Styan's inequality (7) we have

$$R * R^{-1} \geq (R * R^{-1})\nabla I \geq (R * R)^{-1}. \tag{9}$$

The second proof is due to Furuta's inequality [4]:

$$(CA^{-1}C) * (DB^{-1}D) \geq (C * D)(A * B)^{-1}(C * D)$$

for positive invertible operators A, B and positive operators C, D. If we replace $A = B = C = R$ and $D = I$, then since $R * I = I$, we have

$$R * R^{-1} \geq (R * I)(R * R)^{-1}(R * I) = (R * R)^{-1}.$$

For the third proof, note that $R*$ is a unital positive linear map. Hence we see that (8) is the direct consequence of Kadison's Schwarz inequality (cf. [5, Theorem 1.17, (ii)]).

□

Related to (7), R. Nakamoto has observed the following fact:

$$(R * R^{-1})\nabla I \geq (R * R)^{-1/2}, \tag{10}$$

which is induced from (8) and the arithmetic-geometric mean inequality with respect to $(R * R)^{-1}$ and I.

4. In this section, we shall give a few concluding remarks. First we prove

Proposition 6. *If r_{ij} are entries of a correlation operator R, then*

$$|r_{ij}| \leq 1. \tag{11}$$

Proof. Since R is positive, there is a square root $R^{1/2}$ of R, and we have

$$\|R^{1/2}e_k\|^2 = \langle Re_k, e_k\rangle = 1,$$

or $\|R^{1/2}e_k\| = 1$, so that

$$|r_{ij}| = |\langle Re_i, e_j\rangle| = |\langle R^{1/2}e_i, R^{1/2}e_j\rangle| \leq \|R^{1/2}e_i\|\|R^{1/2}e_j\| = 1,$$

which implies (11).

□

From the arithmetic-geometric mean inequality, we have

$$A * I \leq (A^2\nabla I) * I$$

for any positive operator A. A complement of this inequality is given as follows:

Theorem 7. *If A is a positive operator satisfying $mI \leq A \leq MI$ and $k = M\nabla 1/m$ for some positive scalars $m < M$, then*

$$(A^2\nabla I) * I \leq kA * I. \tag{12}$$

Proof. Since

$$A^2\nabla 1 = A^{1/2}(A\nabla A^{-1})A^{1/2} \leq A^{1/2}(M\nabla 1/m)A^{1/2} = kA,$$

we have the desired inequality (12).

□

Finally, we define $A^{[1/2]} = (a_{ij}^{1/2})$ for an operator $A = (a_{ij})$ with $a_{ij} \geq 0$, and ask if $A^{[1/2]} \geq 0$ whenever $A \geq 0$. We can give a negative answer with a 3×3 matrix as follows: Let

$$A = \begin{bmatrix} 1 & a & b \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}, \tag{13}$$

where $a,\ b > 0,\ a + b > 1$ and $a^2 + b^2 \leq 1$. Then $A \geq 0$ since $\det \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix} = 1 - a^2 > 0$ and $\det A = 1 - a^2 - b^2 \geq 0$. But $A^{[1/2]} \not\geq 0$ since $\det A^{[1/2]} = 1 - a - b < 0$.

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Saichi IZUMINO
University of Toyama
Gofuku, Toyama 930-8555, JAPAN
s-izumino@h5.dion.ne.jp

Masahiro NAKAMURA
Department of Mathematics Osaka Kyoiku University
Kashiwara, Osaka 582-8582, JAPAN

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